

Empirical scaling laws for Economic VaR

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1st November 2005 – CCFEA Workshop



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Contribution of the paper

- Using option prices, we can extract implied Risk Neutral Densities (RND), which contain the market expectations of future prices of the underlying.
- Economic-VaR (E-VaR) is an implied risk measure calculated from the implied RNDs.
- The main contribution of this paper is a new methodology to scale E-VaR.

Outline

1. Implied Risk Neutral Densities
2. Value at Risk measures
3. Empirical Scaling law
4. Empirical results

1. **Implied Risk Neutral Densities**

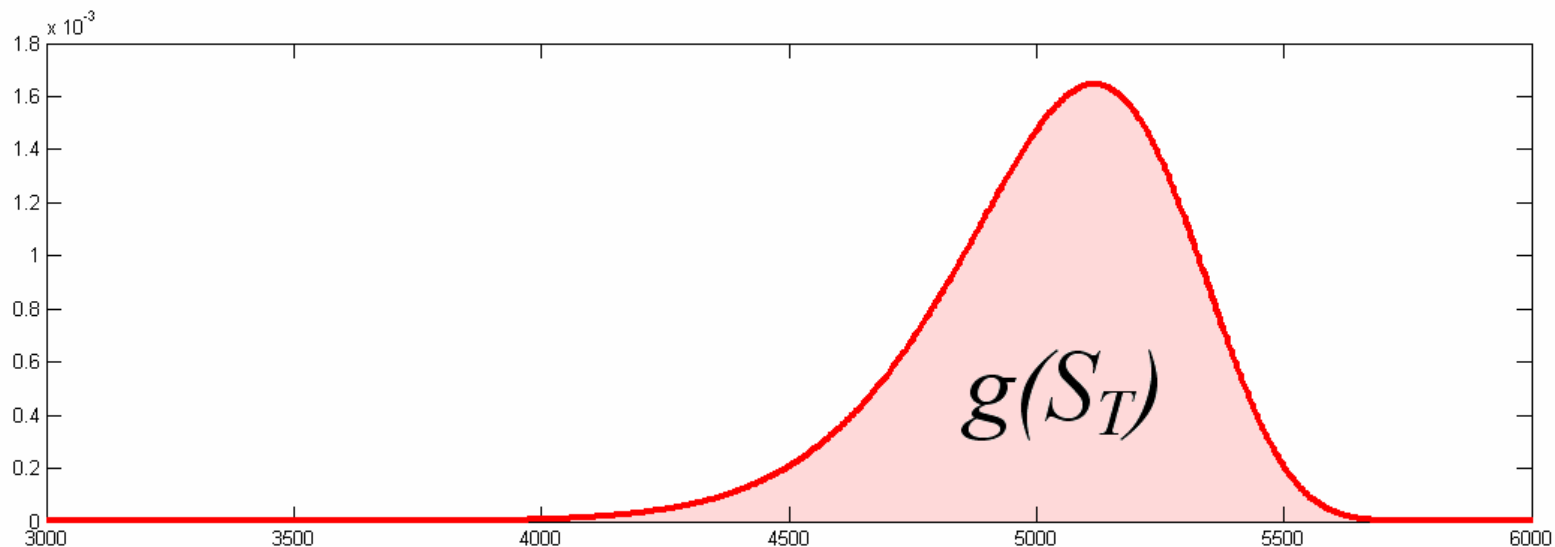
2. Value at Risk measures

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The Risk Neutral Density (RND)

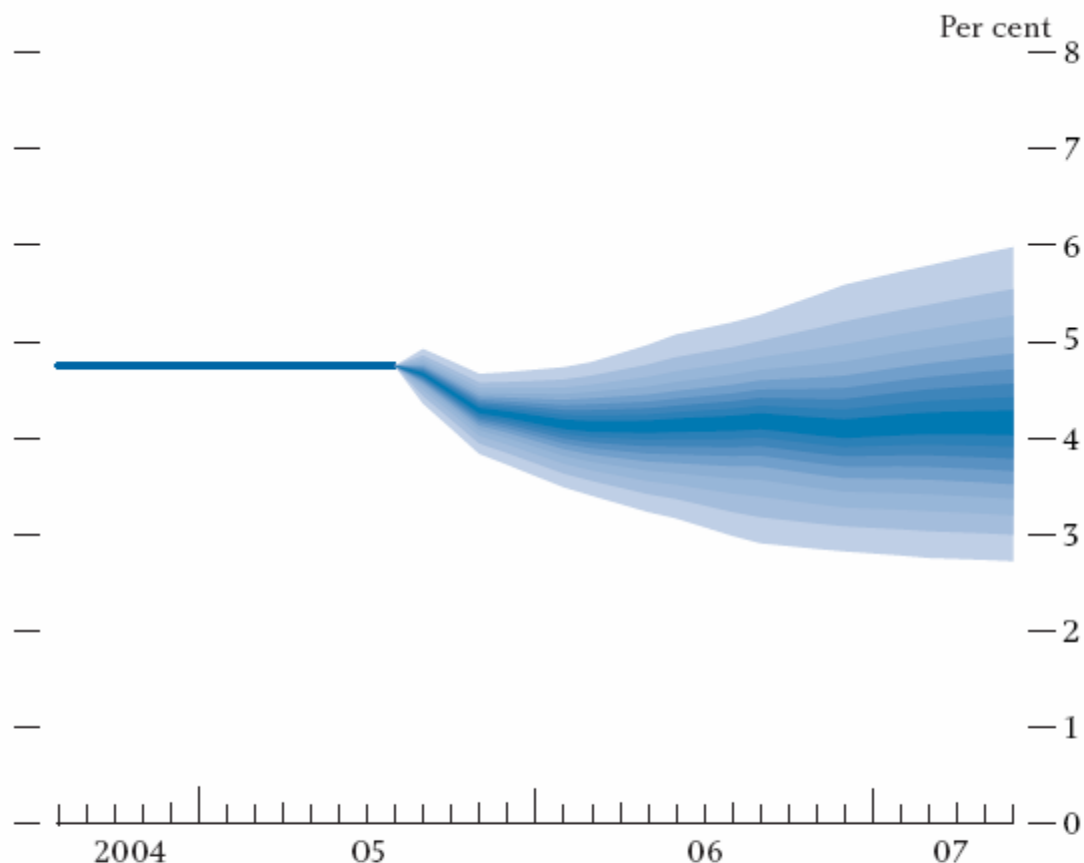
- Using a cross section of traded option prices at different strikes, we can extract the Risk Neutral market expectation of the underlying price distribution at maturity T



But, are implied RNDs useful?

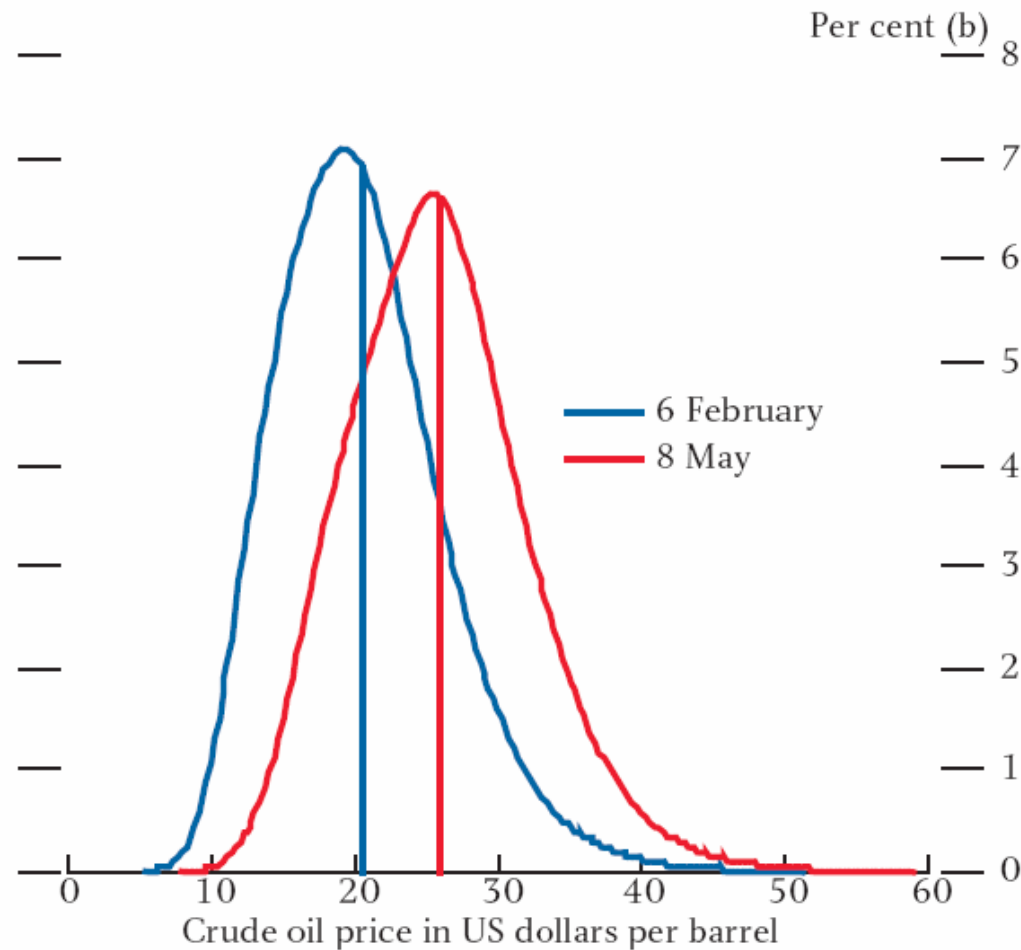
- In every Bank of England Inflation Report, there is the *fan chart* illustrating market beliefs about future interest rates.
- In the Inflation Report, May 2001, changes in the implied distributions of oil price were used to argue that the market uncertainty about oil prices increased considerably since February 2001.
- The Bank of England also published daily estimates of Economic-Value at Risk, for some constant time horizons. See: <http://www.bankofengland.co.uk/statistics/implicitpdfs/>

Market beliefs about future interest rates



The mean of the fan chart is the market rate profile for the fifteen-day average ending 3 August, consistent with the measure of interest rates shown in Table 6.A. The distribution is derived using the prices of options on three-month Libor futures contracts traded on LIFFE. It is constructed by averaging the daily distributions around a common mean for each of the fifteen days. The average is calculated for each probability band at each quarter. The fan chart depicts the probability of outcomes for interest rates in the future. It has a similar interpretation to the fan charts in the Overview and in this section of the *Report*. The chart is only indicative of market expectations because it is based on different, though related, instruments to the Bank's repo contracts, and is estimated on the assumption that investors are risk-neutral.

Likely outcomes for oil prices six months ahead^(a)



Sources: NYMEX and Bank of England.

- (a) Derived from options prices for West Texas Intermediate (WTI) crude oil. Prices of WTI tend to be around \$1 higher than those of Brent crude oil.
- (b) Probability of oil price being between 50 cents above and below any given price.

No-arbitrage European option price

- Following Harrison and Pliska (1981), there exists a risk neutral density (RND) function, $g(S_T)$, such that the call option price can be written as:

$$\begin{aligned} C_t(K) &= E_t^Q \left(e^{-r(T-t)} \max(S_T - K, 0) \right) \\ &= e^{-r(T-t)} \int_K^{\infty} (S_T - K) g(S_T) dS_T \end{aligned}$$

- where $E^Q[\cdot]$ is the risk-neutral expectation operator, conditional on all information available at time t .

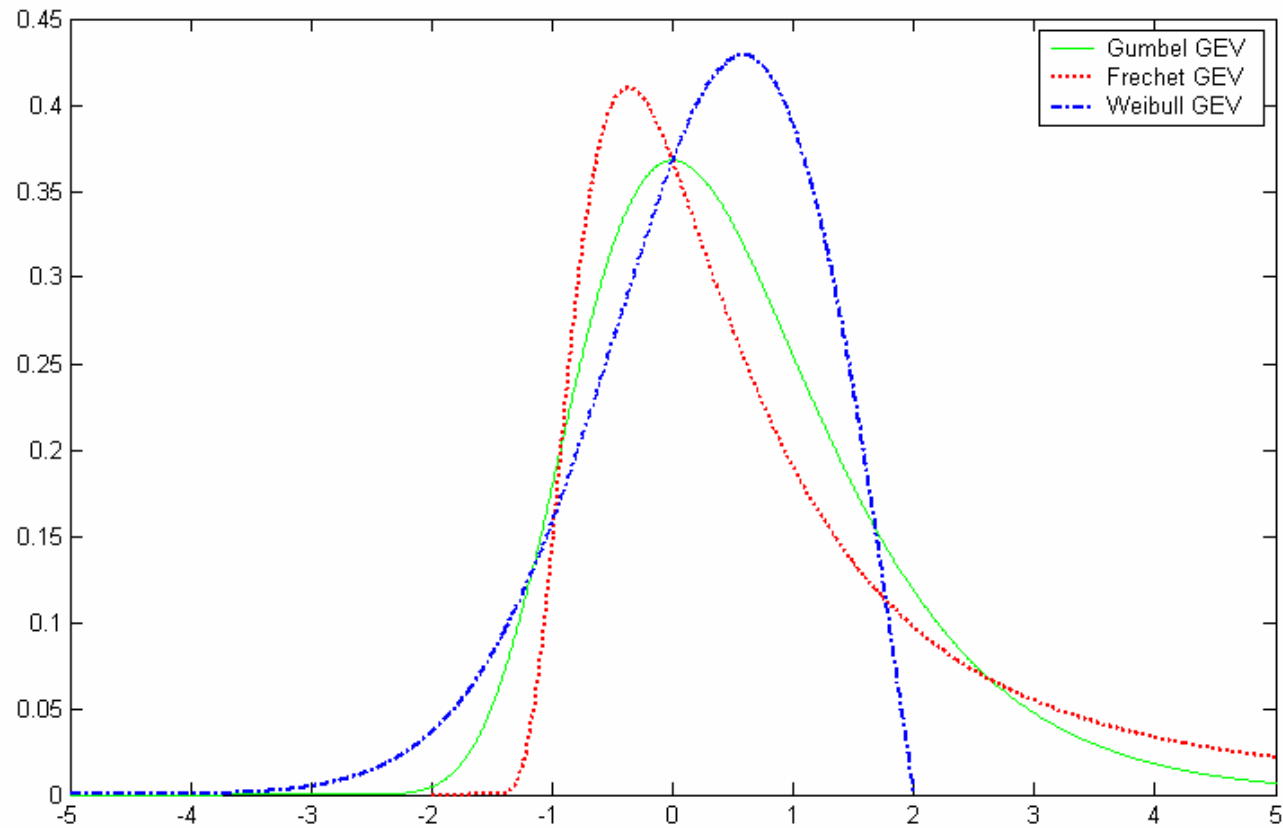
The GEV distribution

- The standardized GEV distribution is given by:

$$G(x) = \begin{cases} \exp\left(-\left(1 + \xi\left(\frac{x - \mu}{\sigma}\right)\right)^{-1/\xi}\right) & \text{for } \xi \neq 0, \text{ with } 1 + \xi\left(\frac{x - \mu}{\sigma}\right) > 0 \\ \exp\left(-\exp\left(-\frac{x - \mu}{\sigma}\right)\right) & \text{for } \xi = 0 \end{cases}$$

- where:
 - μ is the location parameter
 - σ is the scale parameter
 - ξ is the shape parameter

The GEV for different values of ξ



Methodology of RND estimation

- For a given day, we have a set of N traded option prices with the same maturity, but different strikes.
- We use a non-linear least squares algorithm to find the set of parameters that minimize the sum of squared errors:

$$SSE(t) = \min_{\zeta, \mu, \sigma} \sum_{i=1}^N \left(C_t(K_i) - \tilde{C}_t(K_i) \right)^2$$

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Value at Risk

- One of the most popular risk measures in finance.
- It is an estimate, with a given degree of confidence, of how much can be lost from a portfolio over a given time horizon.
- One of the simplest methods to estimate VaR is the Historical method, based in order statistics.
 - The 99% 1-day VaR of a portfolio is the third largest lost in the previous 250 trading days.

Economic VaR (E-VaR)

- Introduced by Ait-Sahalia and Lo (2000), and calculated from an option implied Risk Neutral Density (RND).
- It is a forward looking measure, since the implied RND gives us information about the market expectation of future uncertainty.

The time to maturity effect

- When calculating a time series of E-VaRs we encounter the time to maturity effect.
- Option prices have a fixed maturity.
- As we approach maturity, the time horizon of the implied RND shortens, and densities of consecutive days are not directly comparable.
- We need a methodology to remove this time to maturity effect.

Bank of England methodology

- Currently, the Bank of England calculates E-VaRs from a constant time horizon densities as follows:
 1. Convert option prices to deltas (using B-S)
 2. Interpolate delta surface at a specified maturity
 3. Convert interpolated deltas to option prices
 4. Estimate RND with a non-parametric method
 5. Calculate E-VaR as quantiles of the constant time horizon RND

Drawbacks of this methodology

- Unable to calculate E-VaRs for shorter horizons than the closest maturity, or longer horizons than the longest maturity (very difficult to extrapolate in the delta space)
- Only uses options from two maturities when interpolating in the delta space. So it does not use all option data available.
- Conversions to and from delta space makes this methodology more difficult to both understand and implement.

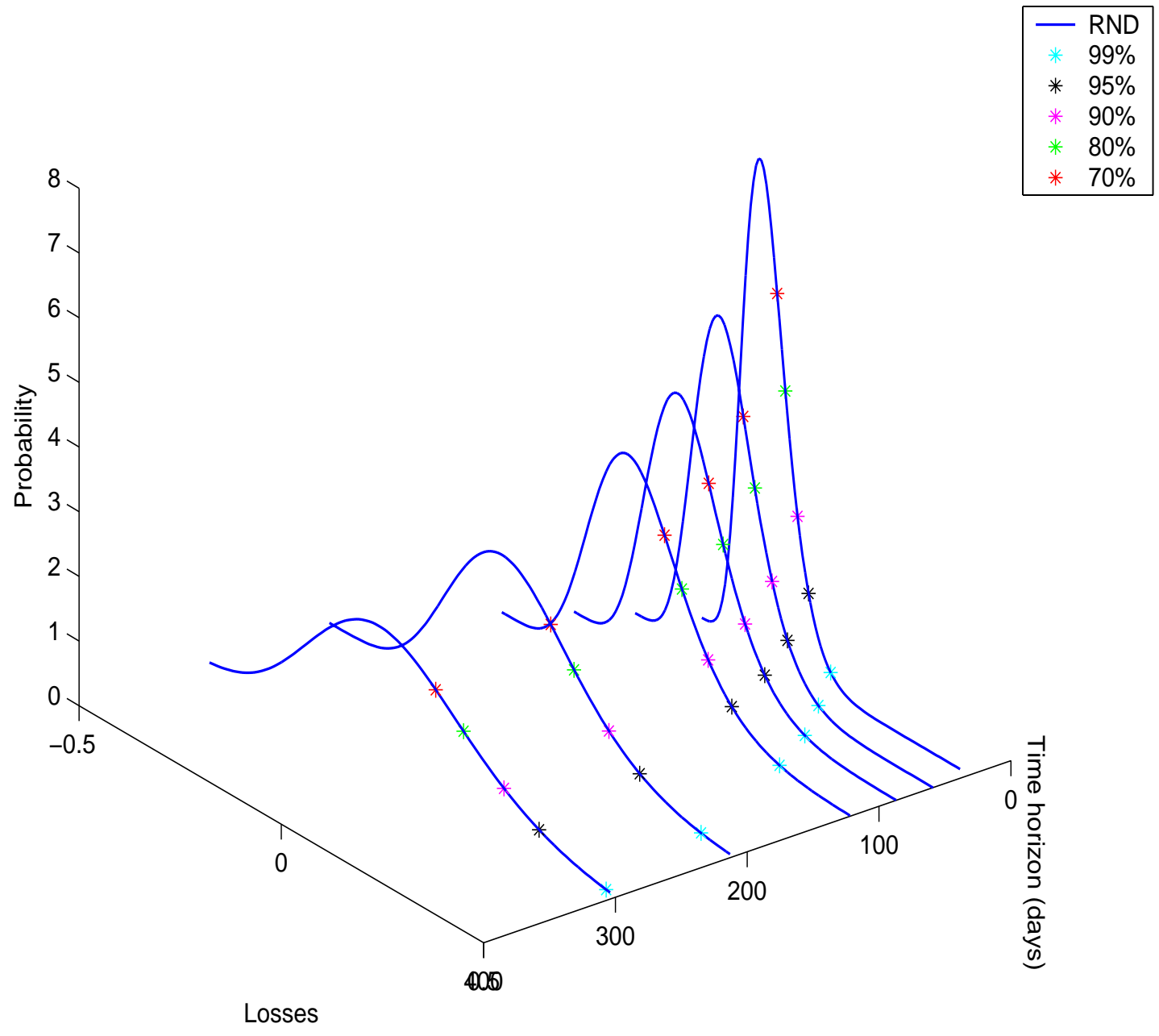
1. Implied Risk Neutral Densities
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Our methodology

- We propose a new methodology for obtaining a constant time horizon E-VaR.
- We use an empirical scaling law:
 1. Estimate the term structure of RNDs
 2. Calculate the quantiles of each RND
 3. Linear regression in the quantile space
 4. Calculate E-VaR values at required horizon

1. Term structure of RNDs

- In most markets, there are option contracts trading for different maturities T_1, T_2, \dots, T_N .
- For example, in the UK, there are traded options with maturities on the closest 3 months, and also quarterly (Mar, Jun, Sep, and Dec).
- At any given day, we can extract from traded option prices a term structure of RNDs.

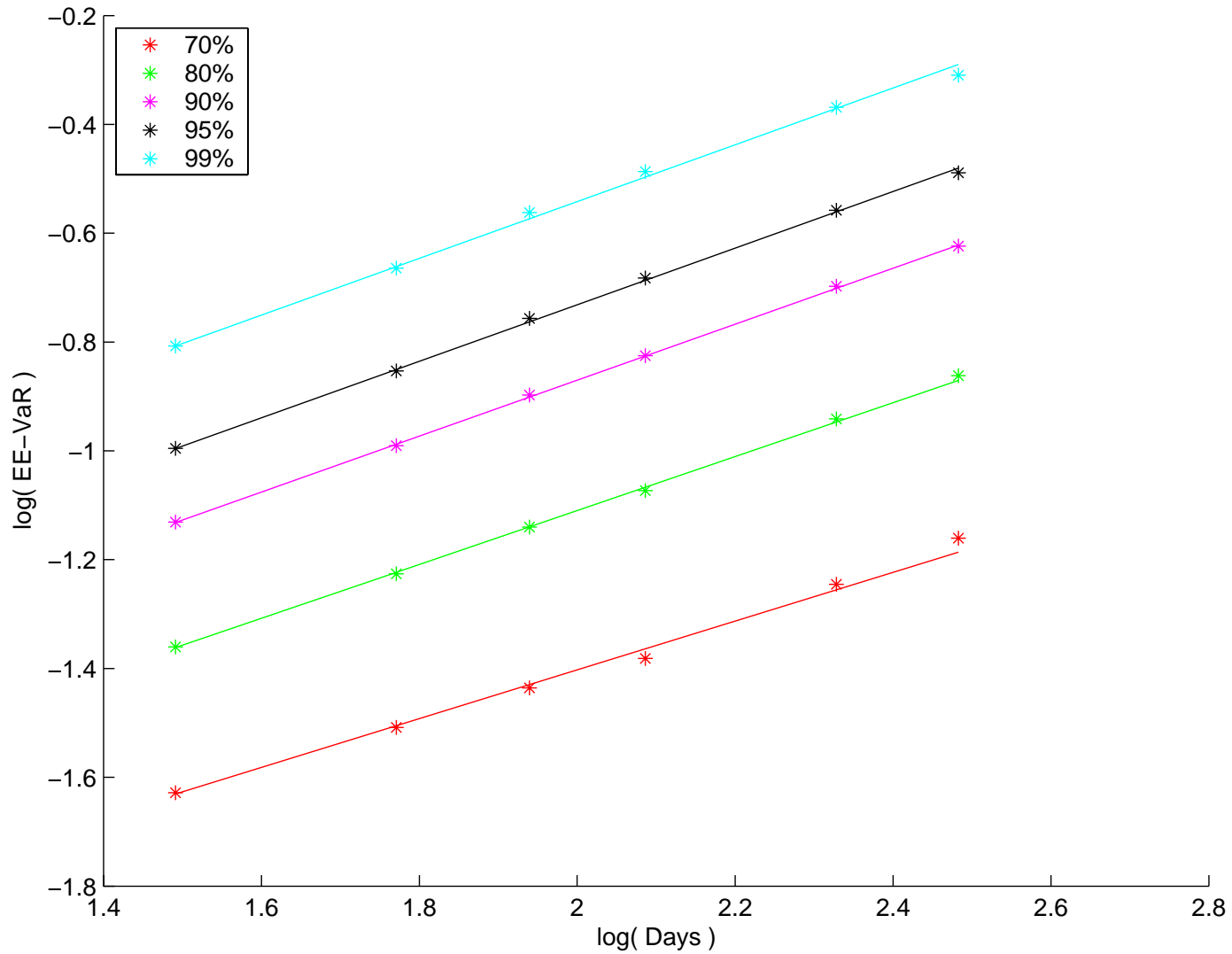


2. Calculate the quantiles

- The E-VaR values are calculated as the quantile of the GEV density at each confidence interval.

Expiry month	Days to maturity	Number options	E-VaR				
			70%	80%	90%	95%	99%
Sep-01	31	44	2.4%	4.4%	7.4%	10.1%	15.6%
Oct-01	59	31	3.1%	5.9%	10.2%	14.0%	21.7%
Nov-01	87	13	3.7%	7.2%	12.7%	17.5%	27.4%
Dec-01	122	16	4.2%	8.5%	15.0%	20.8%	32.6%
Mar-02	213	13	5.7%	11.4%	20.1%	27.7%	42.8%
Jun-02	304	10	6.9%	13.7%	23.8%	32.4%	49.0%

3. Regression in quantile space



4. Calculate E-VaR at any horizon

- The E-VaR values appear to be linear with days to maturity k , in a log-log scale (in average, $R^2 > 0.99$).

$$\log(EVaR(k, cl)) = b \log(k) + c$$

- b : the slope
 - c : the intercept, equal to the log of the 1 day E-VaR
- After having estimated b and c , we can calculate the E-VaR at any horizon k by rewriting the above equation:

$$EVaR(k, cl) = k^b e^c = k^b EVaR(1, cl)$$

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Data description

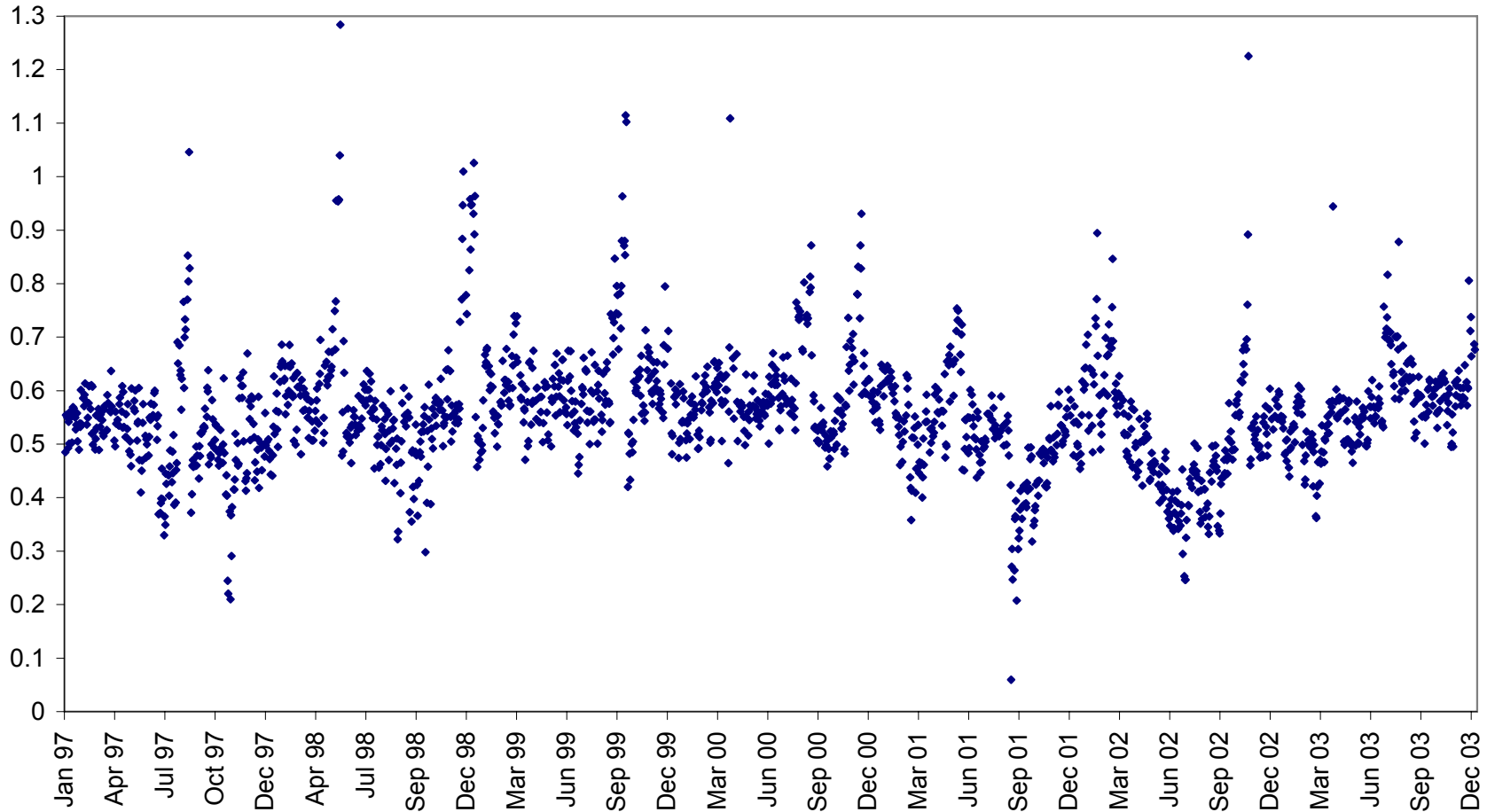
- We used closing prices of traded European options on the FTSE 100 index.
- Period of study Jan0 1997 – Dec 2003
- Interest rate: 3-months LIBOR rate.

Regression results

- The average slope and intercept for each quantile, calculated for each day between 1997-2003.

Confidence level	<i>b</i>	<i>c</i>	1-day EE-VaR = 10^c
70%	0.41 (0.12)	-2.24 (0.26)	0.6%
80%	0.48 (0.09)	-2.04 (0.23)	0.9%
90%	0.51 (0.09)	-1.84 (0.23)	1.4%
95%	0.53 (0.09)	-1.73 (0.23)	1.9%
99%	0.56 (0.11)	-1.57 (0.25)	2.7%
Average	0.50	-1.88	1.5%

Time varying scaling law

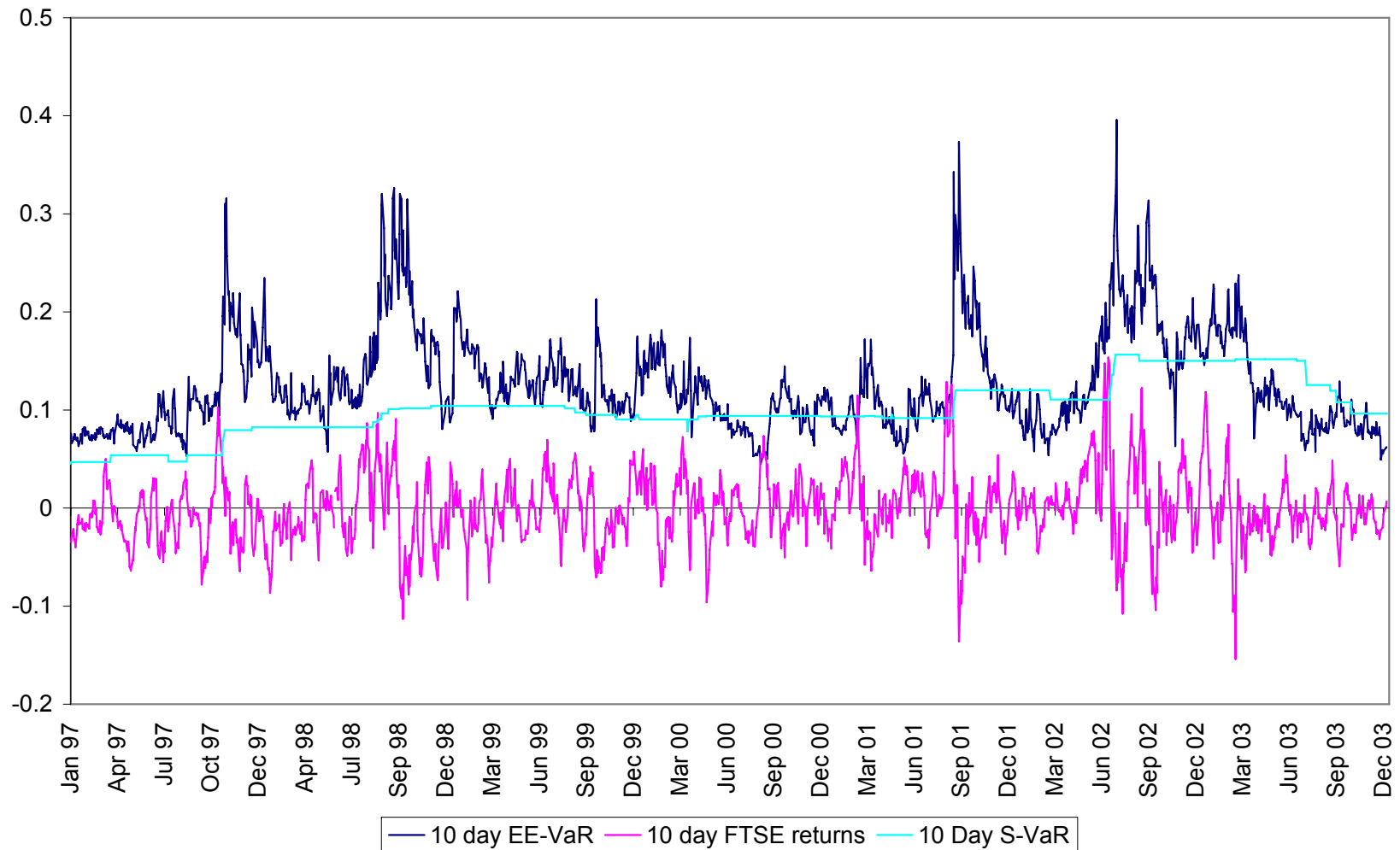


Time series of the b estimates (the slope) at the 99% confidence level.

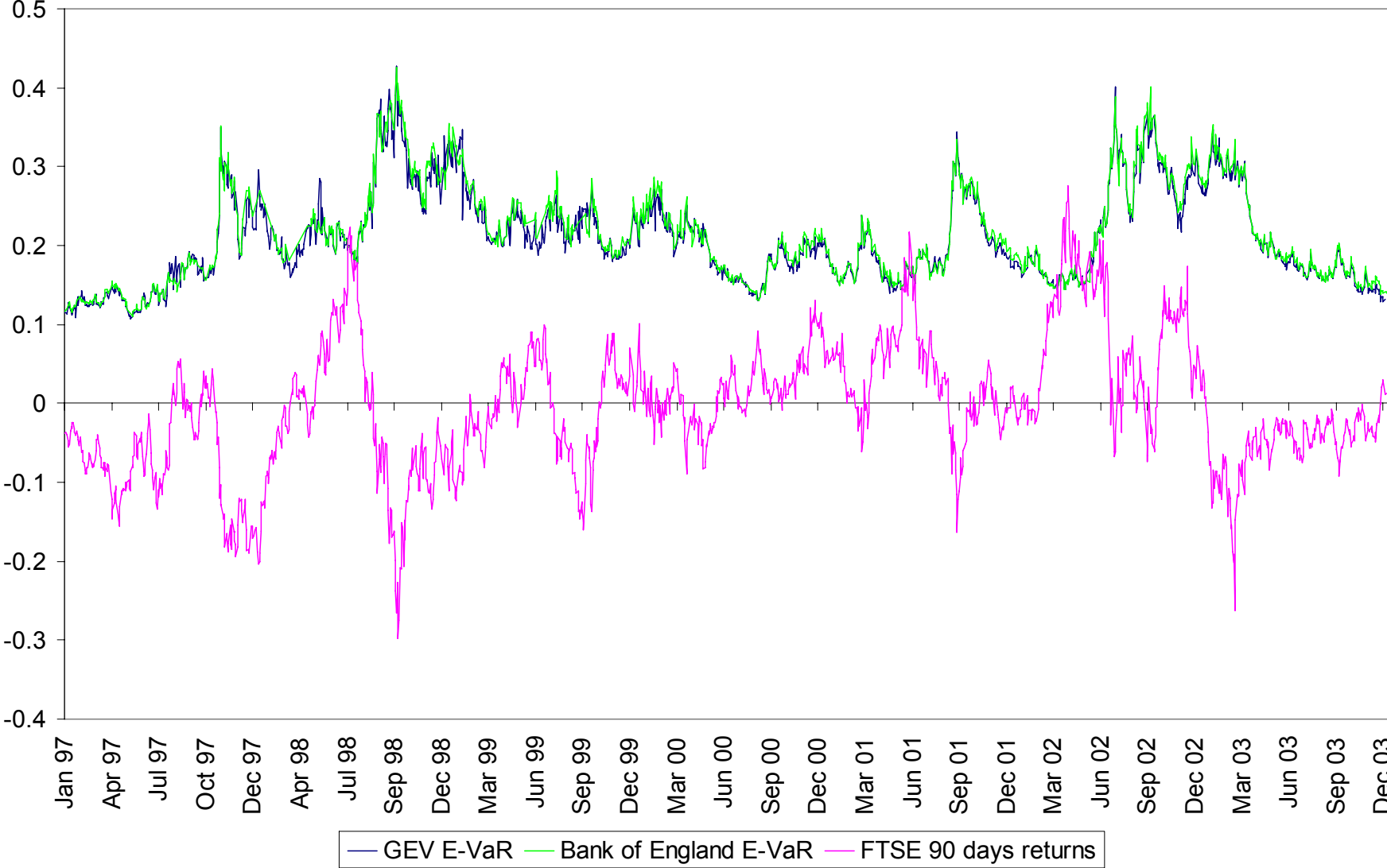
Backtesting performance

- The performance of a VaR methodology is assessed by calculating how close the percentage violations of the predicted VaR value is to the benchmark.
 - For example, at the 99% confidence level, we should exceed the predicted VaR value 1% of the time. If $> 1\%$, we are underestimating risk.

EE-VaR vs. Historical S-VaR



BoE E-VaR vs. EE-VaR @ 95% quantile, 90 days time horizon

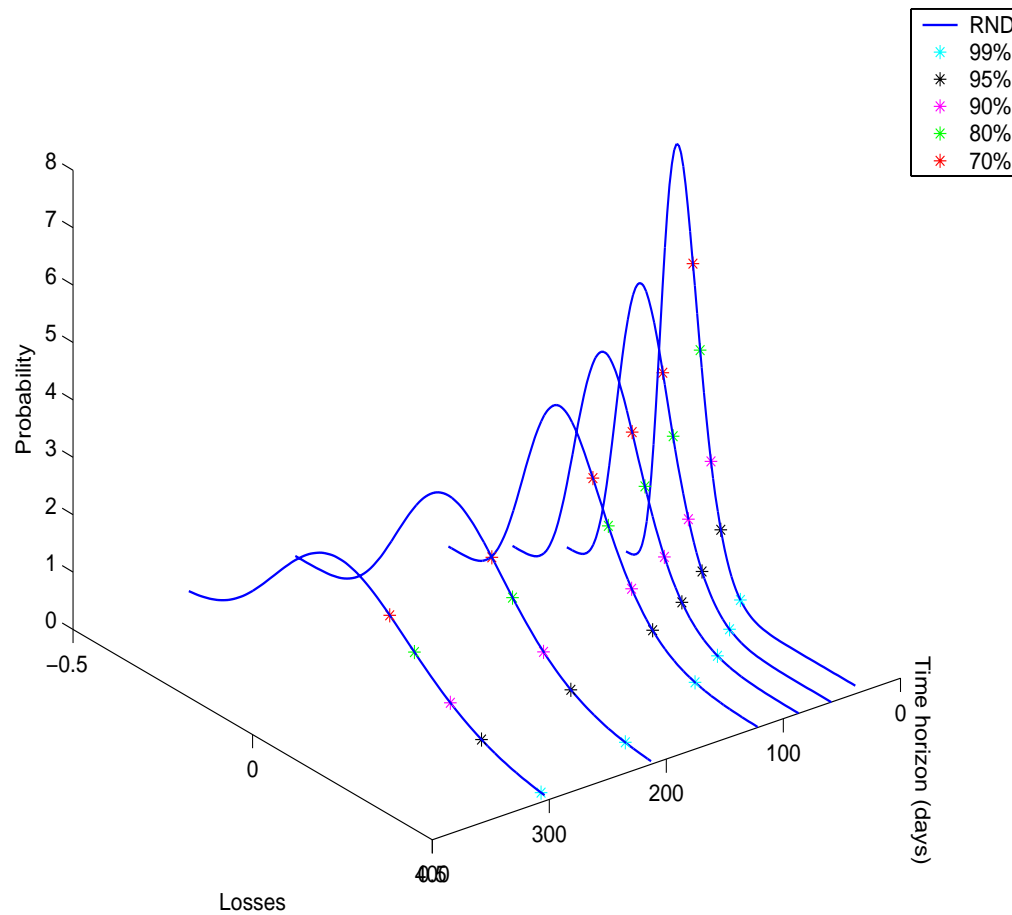


Conclusions

- We have proposed an alternative methodology to calculate Economic-VaR. It is simpler than the currently used by the Bank of England.
- The E-VaR estimates we obtain are very similar to the BoE estimates, even though the methodologies are very different:
 - parametric vs. non-parametric
 - Interpolation in quantile space vs. interpolation in delta space
- Our methodology also delivers an empirical scaling law implied by option prices. This allows us to calculate E-VaR values at any time horizon, including outside maturities available ($T < T_1$ and $T > T_N$)
- However, based on the backtesting performance, we find that E-VaR is too conservative and too volatile to be used as a capital adequacy requirement. We agree with Ait-Sahalia and Lo recommendation of using E-VaR as an additional tool in Risk Management, but not as a replacement of Historical-VaR, since it contains different information.

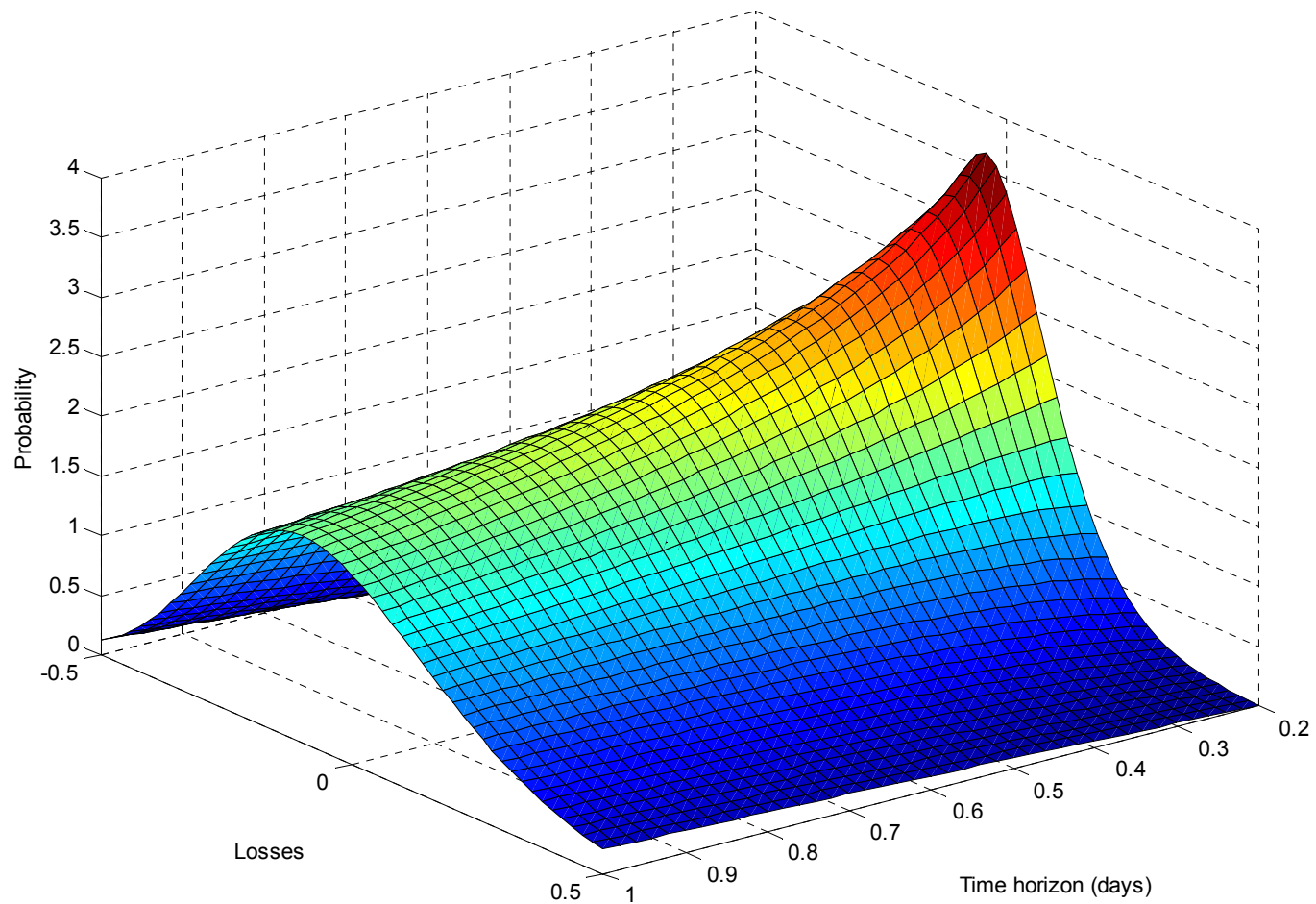
Future work

- Instead of modelling each maturity separately...



Future work

- Model the implied RND process as a function of T .



- At the moment, for a given day, we obtain a different set of parameters for μ, σ, ξ for each maturity available.
- The aim is to write down the GEV distribution function as a function of time, similar to the Black Scholes framework where $R_t \sim N(\mu t, \sigma^2 t)$
- Evidence that standard deviation does not scale with square-root of time.
- Challenge: the GEV moments and GEV parameters do not have a one-to-one relationship.

GEV moments

$$\text{Mean} = \mu + \left[\frac{\Gamma(1-\xi) - 1}{\xi} \right] \sigma$$

$$\text{Variance} = \left[\frac{\Gamma(1-2\xi) - \Gamma^2(1-\xi)}{\xi^2} \right] \sigma^2$$

- How do we scale them with T??